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A note on Mertens' formula for arithmetic progressions

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Abstract

We study the Mertens product over primes in arithmetic progressions, and find a uniform version of previous results.

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1. Introduction

Denote by p a prime number. We are interested in studying a generalization of the famous Mertens formula

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}+\mathcal{O}\left(\frac{1}{\log^2 x}\right)\quad \text{as } x\to+\infty, \tag{1}$$

where γ is the Euler constant. In particular, we consider the primes belonging to arithmetic progressions and obtain a suitable asymptotic formula for the product corresponding to (1), which is uniform in a wide range for the modulus. More precisely, let a, q be integers with $(a,q)=1$,

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and define

$$P(x;q,a)=\prod_{\substack{p\leqslant x\\ p\equiv a \bmod q}}\left(1-\frac{1}{p}\right). \tag{2}$$

We are interested in the range of uniformity in q for the asymptotic formula for the product in (2): such an asymptotic formula will depend on the existence of the "exceptional zero" (or "Siegel zero") for a suitable set of Dirichlet L-functions. We will give an accurate description of this phenomenon later in this Introduction (see Lemma 1 below).

Let $L(x)=\exp((\log x)^{3/5}(\log\log x)^{-1/5})$. For the classical formula in (1), a sharper error term was proved by A.I. Vinogradov [1,2]. At present the best known error term in (1) is of the form $\mathcal{O}(L(x)^{-c})$, where $c>0$ is an absolute constant: see Vasil'kovskaja [3] or, for the line of the proof, pages 80–81 of Prachar's book [4]. An upper bound for the product in (2) has been recently obtained by Bordellès [5].

Considering the arithmetic progressions, Uchiyama [6], Williams [7], Grosswald [8] and Vasil'kovskaja [3] obtained an asymptotic formula for the product in (2) for a fixed arithmetic progression, that is, without any uniformity in q. In particular Williams [7] proved that

$$P(x;q,a)=\frac{C(q,a)}{(\log x)^{1/\varphi(q)}}+\mathcal{O}\left(\frac{1}{(\log x)^{1/\varphi(q)+1}}\right)$$

as $x\to+\infty$, where $C(q,a)$ is real and positive and satisfies

$$C(q,a)^{\varphi(q)}=e^{-\gamma}\frac{q}{\varphi(q)}\prod_{\chi\neq\chi_0}\left(\frac{K(1,\chi)}{L(1,\chi)}\right)^{\overline{\chi}(a)}.$$

Here $L(s,\chi)$ is the Dirichlet L-function associated to the Dirichlet character χ mod q and χ_0 is the principal character to the modulus q. The function K is defined by means of

$$K(s,\chi)=\sum_{n=1}^{+\infty}k_\chi(n)n^{-s}, \tag{3}$$

where $k_\chi(n)$ is the completely multiplicative function whose value at primes is given by

$$k_\chi(p)=p\left(1-\left(1-\frac{\chi(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\chi(p)}\right).$$

Here we improve on Williams' result in three aspects: we insert a new term in the asymptotic expansion of the product defined in (2) improving at the same time the size of the error term and, moreover, our result is uniform in the q-aspect. Furthermore, in Section 6 we prove the following much simpler formula for the value of the constant $C(q,a)$

$$C(q,a)^{\varphi(q)}=e^{-\gamma}\prod_{p}\left(1-\frac{1}{p}\right)^{\alpha(p;q,a)},$$

where $\alpha(p;q,a)=\varphi(q)-1$ if $p\equiv a \bmod q$ and $\alpha(p;q,a)=-1$ otherwise.

We now recall some analytic results on the zero-free region of Dirichlet L-functions to be used in the statement and in the proof of our Theorem 2 below. The proof of the first part can be obtained following the results of Korobov and I.M. Vinogradov on the zero-free region of the Riemann ζ function. See also the notes to §9 of Montgomery [9] and Richert [10]. For the lower bound in the second part see, e.g., §14 of Davenport [11].

Lemma 1. *Assume that $T \geqslant 3$ and $Q \geqslant 1$. There exists a constant $c_1 > 0$ such that $L(\sigma + it, \chi) \neq 0$ whenever*

$$\sigma \geqslant 1 - \frac{c_1}{\log Q + (\log T)^{2/3}(\log\log T)^{1/3}}, \quad |t| \leqslant T,$$

for all the Dirichlet characters χ mod q where $q \leqslant Q$, with the possible exception of at most one primitive character $\tilde{\chi}$ mod $\tilde{r}$ with $\tilde{r} \leqslant Q$. If it exists, the character $\tilde{\chi}$ is real and quadratic and the exceptional zero $\tilde{\beta}$ of $L(s, \tilde{\chi})$ is unique, real, simple and there exists a constant $c_2 > 0$ such that

$$\frac{c_2}{\tilde{r}^{1/2}\log^2 \tilde{r}} \leqslant 1 - \tilde{\beta} \leqslant \frac{c_1}{\log Q + (\log T)^{2/3}(\log\log T)^{1/3}}. \tag{4}$$

For fixed $T \geqslant 3$, $Q \geqslant 1$ and $c_1 > 0$, this lemma defines the exceptional zero $\tilde{\beta} = \tilde{\beta}(Q, T, c_1)$, the exceptional character $\tilde{\chi} = \tilde{\chi}(Q, T, c_1)$ and the exceptional modulus $\tilde{r} = \tilde{r}(Q, T, c_1)$ associated to the set of Dirichlet L-functions $L(s, \chi)$ where χ runs over the set of Dirichlet characters mod q, for every $q \leqslant Q$.

Let us fix some more notation: let

$$R(x) = \exp\bigl((\log x)^{2/5}(\log\log x)^{1/5}\bigr). \tag{5}$$

For $A > 0$ take $Q(x) = R(x)^A$ and $T(x) = L(x)^B$, where $B = B(A)$ is the unique positive solution of the equation $(3/5)^{1/3}u^{5/3} + Au = c_1$, and c_1 is the constant occurring in Lemma 1.

For the sake of simplicity in the statements, we also set

$$G(x; q, \tilde{\beta}) = \begin{cases} \exp\{S(x; \tilde{\beta})\} & \text{if } \tilde{\beta} = \tilde{\beta}(R(x)^A, L(x)^B, c_1), \\ 1 & \text{otherwise,} \end{cases} \tag{6}$$

where

$$S(x, \tilde{\beta}) = -\int_x^{+\infty} \frac{\mathrm{d}t}{t^{2-\tilde{\beta}}\log t}. \tag{7}$$

Now we are ready to state our first result.

Theorem 2. *Let $x \geqslant 3$ be a parameter. For every $A > 0$ there exists a constant $B_1 = B_1(A) > 0$ such that*

$$P(x; q, a) = \frac{C(q, a)}{(\log x)^{1/\varphi(q)}}\bigl(1 + \mathcal{O}\bigl(L(x)^{-B_1}\bigr)\bigr)G(x; q, \tilde{\beta})^{\tilde{\chi}(a)/\varphi(q)}$$

as $x \to +\infty$, uniformly for every $q \leqslant R(x)^A$ and any integer a with $(a,q)=1$. Moreover, the factor G defined in (6) *is* 1 *unless there exists an exceptional zero $\tilde{\beta}$ relative to an exceptional modulus $\tilde{r} \leqslant R(x)^A$ and $\tilde{r} \mid q$, and $\tilde{\chi}$ denotes the exceptional character. Finally, the implicit constant in the error term depends only on the choice of A.*

In some applications one simply gets rid of the exceptional zero by choosing a smaller value for c_1 in Lemma 1. It will be clear from the proof that keeping A fixed and taking c_1 smaller leads to a smaller value for $B = B(A)$ and *a fortiori* for B_1, and therefore to a poorer estimate for the error term. Anyway, it is a widespread belief that exceptional zeros may exist only if $(1-\beta)\log R \to 0$ as $R \to +\infty$ for infinitely many of them.

We remark that it is possible to give a more explicit version of Theorem 2, computing the effect of the exceptional zero.

Corollary 3. *Let x, A and $R(x)$ be defined as in the statement of Theorem* 2*, and assume that there exists an exceptional zero $\tilde{\beta}$ relative to an exceptional modulus $\tilde{r} \leqslant R(x)^A$. For $q \leqslant R(x)^A$ with $\tilde{r} \mid q$ we have*

$$P(x;q,a) = \frac{C(q,a)}{(\log x)^{1/\varphi(q)}}\left(1 + \mathcal{O}_A\left(\frac{(\log\log x)^{16/5}}{(\log x)^{3/5}}\right)\right).$$

We also remark that Theorem 2 contains Williams' result: in fact, for fixed q and sufficiently large x, Lemma 1 implies that q is *not* an exceptional modulus, and therefore the term G is 1.

The same ideas lead to the following stronger statement, that is valid under the assumption of the Generalized Riemann Hypothesis (GRH for brevity).

Theorem 4. *Assume the truth of the GRH. Let $x \geqslant 3$ be a parameter. Then*

$$P(x;q,a) = \frac{C(q,a)}{(\log x)^{1/\varphi(q)}}\bigl(1 + \mathcal{O}\bigl((\log x)x^{-1/2}\bigr)\bigr)$$

as $x \to +\infty$, uniformly for every $q \leqslant x$ and any integer a with $(a,q)=1$. The implicit constant is absolute.

2. Ingredients of the proof

The proof of Theorem 2 is based on the following lemmas.

Lemma 5. *(See Vasil'kovskaja [3].) There exists an absolute constant $c > 0$ such that*

$$\prod_{p \leqslant x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\bigl(1 + \mathcal{O}\bigl(L(x)^{-c}\bigr)\bigr).$$

Proof. See [3] or follow the proof on pages 80–81 of [4] inserting the Korobov [12,13] and Vinogradov [14] zero-free region for the Riemann zeta function. □

Lemma 6. *Let q be a positive integer and χ_0 be the principal character to the modulus q. For $q \leqslant x$ we have*

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\chi_0(p)} = \frac{q}{\varphi(q)}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right).$$

Proof. This follows immediately recalling that $\chi_0(p)=1$ if and only if $p\nmid q$ and that $\chi_0(p)=0$ if and only if $p \mid q$. □

The next lemma is a uniform version (with respect to q) of Eq. (2.4) of Williams [7]. We state it in a form that is suited to our intended application.

Lemma 7. *(See Williams [7].) For every integer $1 \leqslant q \leqslant x$ and every Dirichlet character χ defined to the modulus q, let K be defined by* (3). *We have*

$$\prod_{p\leqslant x}\left(1-\frac{k_\chi(p)}{p}\right)^{-1} = K(1,\chi)\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right).$$

The implicit constant in the error term is absolute: *in particular it is independent from q.*

Proof. The proof is essentially the same as Eq. (2.4) of Williams [7]; one has just to check that the implicit constant in the error term is absolute, and this depends on Eq. (2.2) there. □

The last lemma plays a key role in the final deduction of Theorem 2.

Lemma 8. *For every $A>0$ there exists a constant $B_2 = B_2(A) > 0$ such that*

$$\prod_{p\leqslant x}\left(1-\frac{\chi(p)}{p}\right) = \frac{1}{L(1,\chi)}\left(1+\mathcal{O}\left(L(x)^{-B_2}\right)\right)G(x;q,\tilde{\beta})^{\delta(\chi,\tilde{\chi})}$$

uniformly for all integers $1 \leqslant q \leqslant R(x)^A$, and for every non-principal Dirichlet character χ defined to the modulus q. Here $\tilde{\chi}$ is the exceptional character, as defined in Lemma 1, *associated to the set of the Dirichlet L-functions with modulus $q \leqslant R(x)^A$, and $\delta(\chi,\tilde{\chi})$ is* 1 *if χ is induced by $\tilde{\chi}$, and* 0 *otherwise. The implicit constant in the error term depends only on A.*

Proof. Since

$$L(1,\chi)\prod_{p\leqslant x}\left(1-\frac{\chi(p)}{p}\right) = \prod_{p>x}\left(1-\frac{\chi(p)}{p}\right)^{-1}$$

we have

$$\log\prod_{p>x}\left(1-\frac{\chi(p)}{p}\right)^{-1} = \sum_{p>x}\frac{\chi(p)}{p} + \sum_{p>x}\sum_{m\geqslant 2}\frac{\chi^m(p)}{mp^m} = \sum_{p>x}\frac{\chi(p)}{p} + \mathcal{O}\left(x^{-1}\right).$$

Let

$$\theta(x,\chi)=\sum_{p\leqslant x}\chi(p)\log p.$$

Inserting Lemma 1 in the argument on p. 122, §20 of Davenport [11], we have

$$\theta(x,\chi)=-\frac{x^{\tilde\beta}}{\tilde\beta}\delta(\chi,\tilde\chi)+\mathcal{O}\bigl(x(\log x)^2L(x)^{-B(A)}\bigr) \tag{8}$$

uniformly for $q\leqslant R(x)^A$ with any fixed $A>0$, where $B(A)$ is a positive constant depending only on A. Here $\tilde\beta$ is the possible exceptional zero relative to the moduli $\leqslant R(x)^A$, and the corresponding term occurs if and only if the character χ is induced by $\tilde\chi$. Actually, to prove (8) one just takes $T(x)=L(x)^B$ in Eq. (6) on p. 122 of [11] where $B=B(A)$ is the unique positive solution of the equation $(3/5)^{1/3}u^{5/3}+Au=c_1$, and c_1 is the constant occurring in Lemma 1.

By partial summation, we see that

$$\begin{aligned}\sum_{x<p\leqslant y}\frac{\chi(p)}{p}&=\sum_{x<p\leqslant y}\frac{\chi(p)\log p}{p\log p}\\&=\frac{\theta(y,\chi)}{y\log y}-\frac{\theta(x,\chi)}{x\log x}+\int_x^y\theta(t,\chi)\frac{\log t+1}{t^2(\log t)^2}\,\mathrm{d}t.\end{aligned}$$

Letting $y\to+\infty$ and using (8) to ensure the convergence of the improper integral, we obtain that

$$\sum_{p>x}\frac{\chi(p)}{p}=-\frac{\theta(x,\chi)}{x\log x}+\int_x^{+\infty}\theta(t,\chi)\frac{\log t+1}{t^2(\log t)^2}\,\mathrm{d}t. \tag{9}$$

It is easy to see that the contribution to (9) of the term $x(\log x)^2L(x)^{-B(A)}$ in (8) is $\ll L(x)^{-B_2}$ for any fixed positive $B_2<B(A)$. The contribution of the term arising from the possible exceptional zero is

$$\frac{x^{\tilde\beta}}{\tilde\beta x\log x}+\frac{1}{\tilde\beta}\int_x^{+\infty}t^{\tilde\beta}\frac{\mathrm{d}}{\mathrm{d}t}(t\log t)^{-1}\,\mathrm{d}t=-\int_x^{+\infty}\frac{\mathrm{d}t}{t^{2-\tilde\beta}\log t}$$

as one can readily check by means of an integration by parts. The proof is complete comparing the last identity with the definition of $S(x;\tilde\beta)$ in (7). □

3. Proof of Theorem 2

We follow the line of Williams [7]. Recalling the orthogonality relation

$$\sum_{\chi}\chi(p)\overline{\chi}(a)=\begin{cases}\varphi(q) & \text{if } p\equiv a \bmod q,\\ 0 & \text{otherwise,}\end{cases}$$

we have

$$P(x;q,a)^{\varphi(q)} = \prod_{\chi \bmod q} \left(\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^{\chi(p)} \right)^{\overline{\chi}(a)}. \tag{10}$$

This is Eq. (3.1) of Williams. For $\chi = \chi_0$, by Lemmas 5 and 6, we get

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^{\chi_0(p)} = \frac{q}{\varphi(q)} \frac{e^{-\gamma}}{\log x} \bigl(1 + \mathcal{O}\bigl(L(x)^{-c}\bigr)\bigr). \tag{11}$$

For $\chi \neq \chi_0$, Lemmas 7 and 8 imply that

$$\begin{aligned} \prod_{p \leqslant x} \left(1 - \frac{1}{p}\right)^{\chi(p)} &= \prod_{p \leqslant x} \left(1 - \frac{\chi(p)}{p}\right) \prod_{p \leqslant x} \left(1 - \frac{k_\chi(p)}{p}\right)^{-1} \\ &= \frac{K(1,\chi)}{L(1,\chi)} \bigl(1 + \mathcal{O}\bigl(L(x)^{-B_2}\bigr)\bigr) G(x;q,\tilde{\beta})^{\delta(\chi,\tilde{\chi})}. \end{aligned} \tag{12}$$

Notice that the term $G(x;q,\tilde{\beta})$ has a positive exponent if and only if χ is a real character induced by the exceptional character $\tilde{\chi}$: this may happen only if q is a multiple of $\tilde{r}$, and for at most *one* character modulo q. Collecting (11) and (12) and comparing with (10), we see that, for $B_1 = \min(c, B_2)$, we have

$$\begin{aligned} P(x;q,a)^{\varphi(q)} &= \frac{q}{\varphi(q)} \frac{e^{-\gamma}}{\log x} \bigl(1 + \mathcal{O}\bigl(L(x)^{-c}\bigr)\bigr) \\ &\quad \times \prod_{\chi \neq \chi_0} \left\{ \frac{K(1,\chi)}{L(1,\chi)} \bigl(1 + \mathcal{O}\bigl(L(x)^{-B_1}\bigr)\bigr) G(x;q,\tilde{\beta})^{\delta(\chi,\tilde{\chi})} \right\}^{\overline{\chi}(a)} \\ &= \frac{C(q;a)^{\varphi(q)}}{\log x} \bigl(1 + \mathcal{O}\bigl(L(x)^{-B_1}\bigr)\bigr)^{\varphi(q)} G(x;q,\tilde{\beta})^{\tilde{\chi}(a)}. \end{aligned}$$

The exponent of the term $G(x;q,\tilde{\beta})$ is $\tilde{\chi}(a)$ because, by the remark above, $\tilde{\chi}$ induces at most one character χ mod q, if any, which is real. Since $(a,q) = 1$, we have $\chi(a) = \tilde{\chi}(a)$. Our main result follows immediately.

4. Proof of the corollary

First of all we remark that, defining

$$\mathrm{li}(x) = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{x} \frac{\mathrm{d}t}{\log t},$$

where $x \in \mathbb{R}^+ \setminus \{1\}$, it is possible to express the quantity $S(x;\tilde{\beta})$ defined in (7) as $\mathrm{li}(x^{\tilde{\beta}-1})$, making the change of variables $t = u^{-1/(1-\tilde{\beta})}$. Moreover, it is easy to see that the worst case for the term containing G in Theorem 2 occurs when $q = \tilde{r}$, so it is enough to bound the quantity

$$G(x;\tilde{r},\tilde{\beta})^{\tilde{\chi}(a)/\varphi(\tilde{r})} = \exp\left(\frac{\tilde{\chi}(a)}{\varphi(\tilde{r})}\,\mathrm{li}\left(x^{\tilde{\beta}-1}\right)\right). \tag{13}$$

Furthermore, we also notice that

$$\mathrm{li}(x) = \frac{x}{\log x} + \int_0^x \frac{\mathrm{d}t}{(\log t)^2},$$

and hence we see that, for $x \in (0,1)$, we have

$$\frac{x}{\log x} < \mathrm{li}(x) < 0.$$

Using (13) and the last inequality, we will prove that

$$\left|\frac{\tilde{\chi}(a)}{\varphi(\tilde{r})}\,\mathrm{li}\left(x^{\tilde{\beta}-1}\right)\right| \leqslant \frac{1}{\varphi(\tilde{r})}\frac{x^{\tilde{\beta}-1}}{(1-\tilde{\beta})\log x} \ll_A \frac{(\log\log x)^{16/5}}{(\log x)^{3/5}}. \tag{14}$$

To achieve this goal, we first remark that $x^{\tilde{\beta}-1} \leqslant 1$ and that $\varphi(\tilde{r}) \geqslant C\tilde{r}/\log\log\tilde{r}$ for a suitable positive constant C. By (4) we have that

$$(1-\tilde{\beta})\tilde{r}^{1/2} \geqslant \frac{c_2}{(\log\tilde{r})^2} \geqslant \frac{c_2}{A^2}\frac{1}{(\log x)^{4/5}(\log\log x)^{2/5}}$$

showing that

$$\frac{1}{\varphi(\tilde{r})}\frac{x^{\tilde{\beta}-1}}{(1-\tilde{\beta})\log x} \leqslant \frac{A^2}{Cc_2}\frac{\log\log\tilde{r}}{\tilde{r}^{1/2}}\frac{(\log\log x)^{2/5}}{(\log x)^{1/5}}. \tag{15}$$

We now use the fact that $\tilde{r} \leqslant R(x)^A$ to give the bound $\log\log\tilde{r} \ll_A \log\log x$. Using (4) again, we see that

$$\tilde{r}^{1/2}(\log\tilde{r})^2 \geqslant \frac{c_2}{c_1}\log R(x)^A = \frac{c_2 A}{c_1}(\log x)^{2/5}(\log\log x)^{1/5}$$

so that $\tilde{r}^{1/2} \gg_A (\log x)^{2/5}(\log\log x)^{-9/5}$. Inserting this last estimate in (15), we finally get that (14) holds. Hence the corollary follows since $\exp(u) = 1 + \mathcal{O}(u)$ for $u \to 0$.

5. Proof of Theorem 4

We now assume the validity of GRH. It is clear that both Lemmas 5 and 8 can be improved and that Lemma 1 becomes void. It is also clear that Theorem 4 is an immediate consequence of the following lemma.

Lemma 9. *Assume the truth of the GRH. Uniformly for all characters* $\chi \bmod q$ *with* $q \leqslant x$ *we have the following estimates*

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}\bigl(1+\mathcal{O}\bigl((\log x)x^{-1/2}\bigr)\bigr), \tag{16}$$

$$\prod_{p\leqslant x}\left(1-\frac{\chi(p)}{p}\right)=\frac{1}{L(1,\chi)}\bigl(1+\mathcal{O}\bigl((\log x)x^{-1/2}\bigr)\bigr). \tag{17}$$

Proof. Writing $\pi(x)=\mathrm{li}(x)+E(x)$ with $E(x)\ll x^{1/2}\log x$, by partial summation we easily deduce that there exists a positive constant C such that

$$\sum_{p\leqslant x}\frac{1}{p}=\log\log x+C+\mathcal{O}\bigl((\log x)x^{-1/2}\bigr)$$

and (16) follows by exponentiation.

The proof of (17) is essentially the same as the proof of Lemma 8, using the bound $\theta(x,\chi)\ll x^{1/2}(\log x)^2$ for $q\leqslant x$, as in §20 of Davenport [11]. □

6. An alternative expression for the constant $C(q,a)$

We notice that the argument in Section 3 may be arranged in a different fashion that provides an alternative, simpler form for the constant $C(q,a)$ occurring in the statement of our Theorem 2, showing that the quantity whose $\varphi(q)$th root we take is indeed real and positive. We assume, as we may, that q and a are fixed, with $(a,q)=1$ and that χ is a non-principal character modulo q. Let

$$P(x;\chi)=\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\chi(p)}.$$

Using Eq. (3.2) in Williams [7] we see that

$$\frac{K(1,\chi)}{L(1,\chi)}=\lim_{x\to+\infty}P(x;\chi).$$

Since there is a finite number of Dirichlet characters modulo q, we deduce that

$$C(q,a)^{\varphi(q)}=e^{-\gamma}\frac{q}{\varphi(q)}\lim_{x\to+\infty}\prod_{\chi\neq\chi_0}P(x;\chi)^{\overline{\chi}(a)}.$$

Using again the orthogonality of the Dirichlet characters, we find that

$$\prod_{\chi\neq\chi_0} P(x;\chi)^{\overline{\chi}(a)} = \prod_{p\leqslant x}\prod_{\chi\neq\chi_0}\left(1-\frac{1}{p}\right)^{\chi(pa^{-1})} = \prod_{\substack{p\leqslant x\\ p\nmid q}}\left(1-\frac{1}{p}\right)^{\alpha(p;q,a)}$$

where $\alpha(p;q,a)=\varphi(q)-1$ if $p\equiv a \bmod q$ and $\alpha(p;q,a)=-1$ otherwise. This shows that the right-hand side of the identity defining $C(q,a)$ is real and positive, and therefore that we may safely take its $\varphi(q)$th root. We notice that

$$\frac{q}{\varphi(q)} = \prod_{p\mid q}\left(1-\frac{1}{p}\right)^{-1}.$$

Therefore, if we assume as we may that $x\geqslant q$, we have

$$C(q,a)^{\varphi(q)} = e^{-\gamma}\lim_{x\to+\infty}\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)^{\alpha(p;q,a)} = e^{-\gamma}\prod_{p}\left(1-\frac{1}{p}\right)^{\alpha(p;q,a)}.$$

The infinite product converges, though not absolutely, by the Prime Number Theorem for Arithmetic Progressions.

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